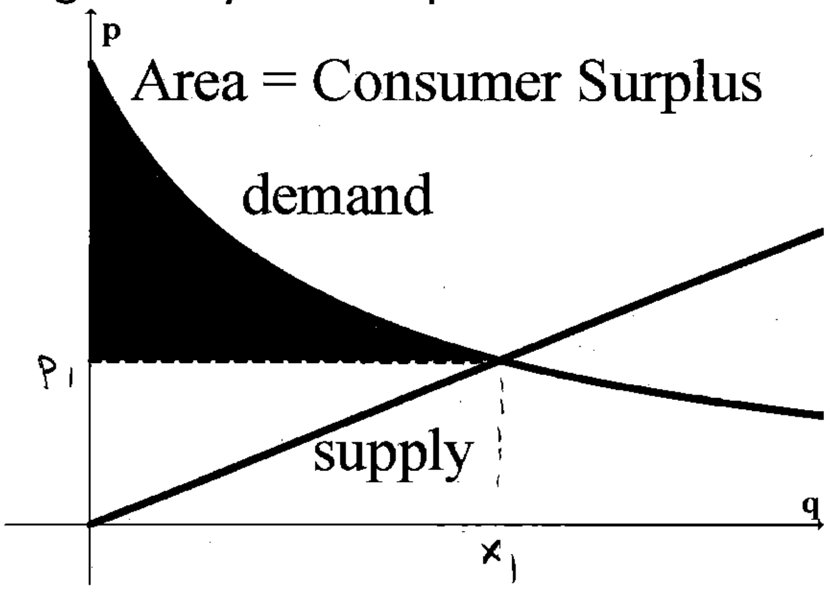
Closing Tues: HW 13.4, 14.1
Closing Thur: HW 14.2 (part 1)

13.4 Consumer/Supplier Surplus *(continued)*

Recall from last time: *Consumer Surplus* is given by the shaped area below:

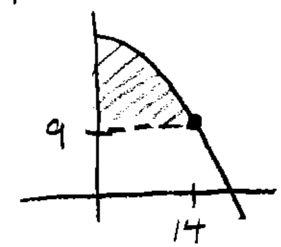


If demand is given by $p = f(x)$ and equilibrium is at $(x, p) = (x_1, p_1)$, then

$$\text{CS} = \int_0^{x_1} f(x)dx - p_1 x_1$$

Example: (HW 13.4/4)
The demand function is $p = 205 - x^2$.
If the equilibrium price is \$9.00 per item, then what is consumer surplus?

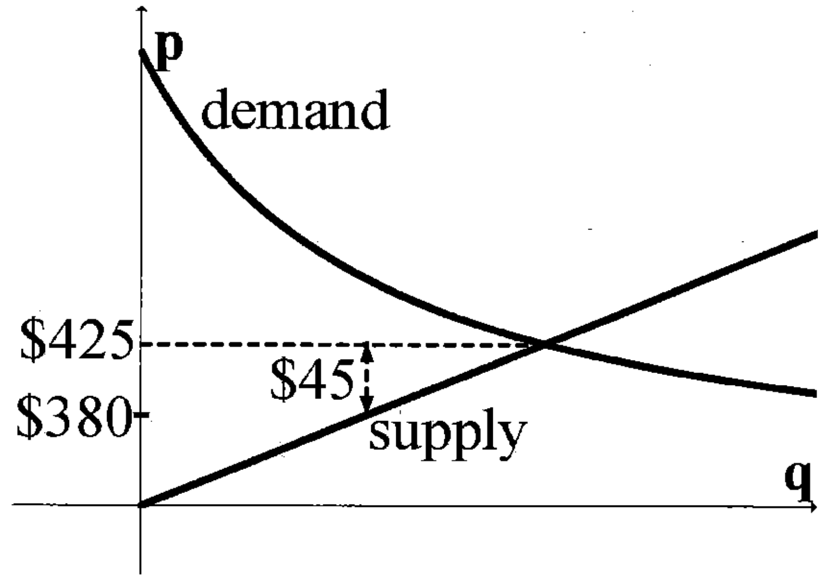
$p_1 = 9 \Rightarrow 9 = 205 - x^2$
$\Rightarrow x^2 = 196$
$\Rightarrow x = 14 \leftarrow x_1$



$$CS = \int_0^{14} 205 - x^2 dx - 9 \cdot 14$$
$$= 205x - \tfrac{1}{3}x^3 \Big|_0^{14} - 126$$
$$= \left(205(14) - \tfrac{1}{3}(14)^3\right) - (0) - 126$$
$$= 1955.33 - 126$$
$$= \boxed{\$1829.33}$$

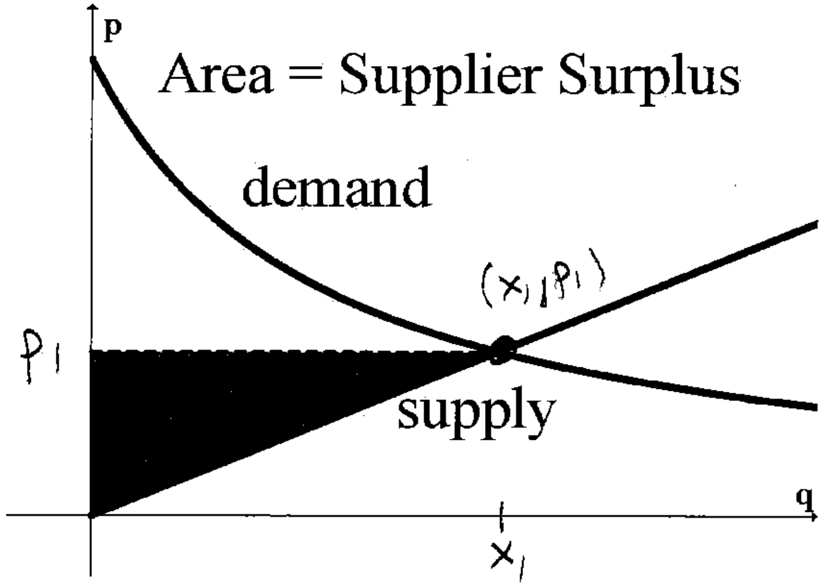
Producer (Supplier) Surplus

Idea: Assume a supplier produced and sold the same bit of technology from my earlier story. They had planned to sell it for \$380 at a different store and as a result had produced fewer quantities than most manufacturers (or you could say they were willing to sell for less than market equilibrium).



But it turned out that the equilibrium price is \$425 and they can sell for a \$45 *surplus* from what they had originally planned (or you could say they left \$45 per item "on the table").

The sum of all moneys for suppliers willing to sell for less than *equilibrium* for a given product is called **Producer Surplus**. It is given by the area of the region below:



If supply is given by $p = g(x)$ and equilibrium is at $(x, p) = (x_1, p_1)$, then

$$\text{PS} = p_1 x_1 - \int_0^{x_1} g(x)dx$$

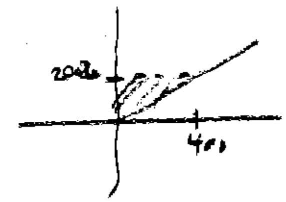
Example: If the supply curve is $p = 2 + 0.5x$, and if market equilibrium is at $(x, p) = (400{,}202)$ then find supplier surplus.

$$\underbrace{202 \cdot 400}_{TR(400)} - \int_0^{400} 2 + 0.5x\, dx$$

$$= \$80{,}800 - \Big[\underbrace{2x + 0.25x^2 \Big|_0^{400}}\Big]$$

$$= 80{,}800 - 40{,}800$$

$$= \boxed{\$40{,}000}$$



Example (Problems 7 and 11 from HW)

Given Demand: $p = \frac{48}{x+2}$

Supply: $p = 3 + 0.1x$

Find consumer and producer surplus under pure competition (meaning at market equilibrium).

Step 1: Find market equilibrium.

$$\textit{Step 2}: CS = \int_0^{x_1} f(x)dx - p_1x_1$$

$$\textit{Step 3}: PS = p_1x_1 - \int_0^{x_1} g(x)dx$$

STEP 1 $\quad 3 + 0.1x = \frac{48}{x+2}$

$(x+2)(3+0.1x) = 48$

$\Rightarrow \quad 0.1x^2 + 3x + 0.2x + 6 = 48$

$0.1x^2 + 3.2x - 42 = 0$

$x^2 + 32x - 420 = 0$

QUAD FORMULA or $(x-10)(x+42) = 0$

$\boxed{x = 10} \quad x = -42$

$x = 10 \Rightarrow p = 3 + 0.1(10) = 4$ ✓

$p = \frac{48}{(10+2)} = 4$ ✓

$\boxed{x_1 = 10, \; p_1 = 4 \quad \text{MARKET EQUILIBRIUM}}$

$$CS = \int_0^{10} \frac{48}{x+2}dx - 4 \cdot 10$$

$$= 48\ln(x+2)\Big|_0^{10} - 40$$

$$= 48\ln(12) - 48\ln(2) - 40$$

$$\approx 86.00445452 - 40$$

$$= \boxed{\$46.00}$$

$$PS = 4 \cdot 10 - \int_0^{10} 3 + 0.1x \, dx$$

$$= 40 - \left[3x + 0.05x^2\Big|_0^{10}\right]$$

$$= 40 - [(3(10) + 0.05(10)^2) - 0]$$

$$= 40 - 35$$

$$= \boxed{\$5}$$

14.1 Multivariable Functions

Up to now, we've been investigating functions that have ***only one*** input.

Examples: TR(q), MC(q), D(t), f(x), etc...

A ***multivariable*** function has more than one input.

Examples:

$C(h, p, x, y, z) = \text{course percentage}$

$A(P, r, t) = Pe^{rt}$

$BMI(w, h) = \dfrac{703w}{h^2}$

$TC(x, y) = 3x + 5y + 10$

$TR(x, y) = 8x + 6y$

Goal: To find and interpret derivatives of multivariable functions. And use them to find critical points.

Idea: Look at one variable at a time.

Ex) $z = \dfrac{x^2}{y^3}$

If $x = 2$, then

$z = \dfrac{4}{y^3} = 4y^{-3}$

$\dfrac{dz}{dy} = -12y^{-4}$

If $y = 2$, then

$z = \dfrac{x^2}{8} = \dfrac{1}{8}x^2$

$\dfrac{dz}{dx} = \dfrac{1}{4}x$

Aside: (You don't need to know this for this course, but I think it might help you visualize what is going on).

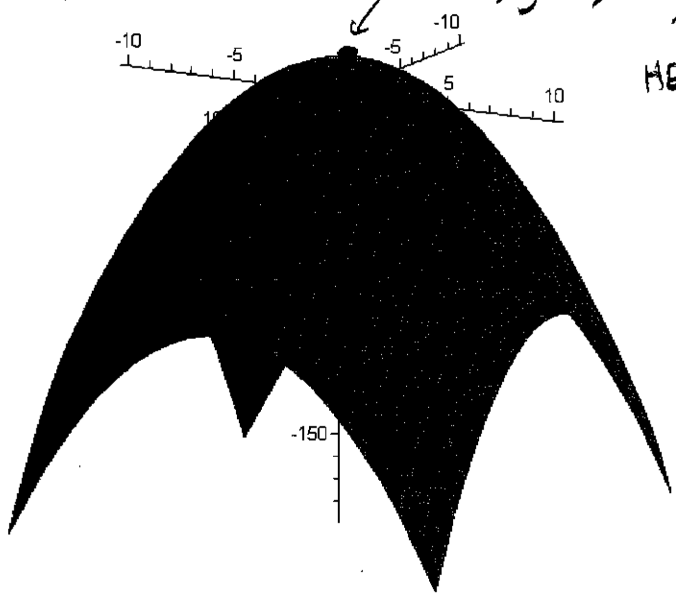
The graph of a 2-variable function is a *surface*, where the output is the height of the surface.

Example: (A "paraboloid")

$z = f(x,y) = 15 - x^2 - y^2$

$x=0, y=0, z=15$

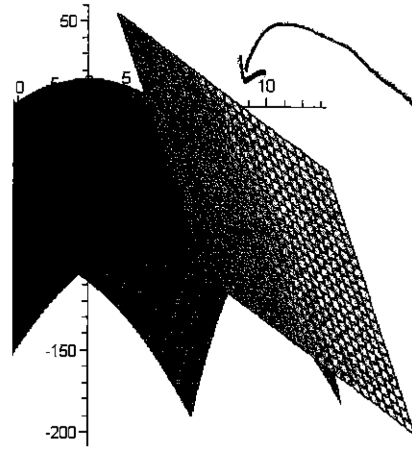
HEIGHT



Example: (A "plane")

$$z = g(x,y) = \underbrace{-14x - 8y + 80}$$

A plane

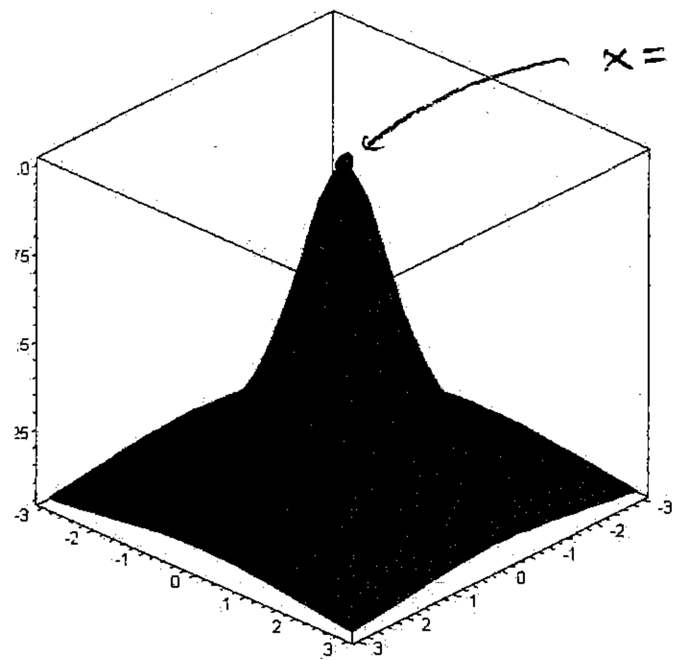


Example: $z = h(x,y) = \dfrac{1}{1+x^2+y^2}$

$x=0,\ y=0$

$z = \dfrac{1}{1+0^2+0^2} = 1$

height



Example:

$$f(x,y) = 4xy + y^2 - 3x - 5y$$

Find and simplify

$$\frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{[4(x+h)y + y^2 - 3(x+h) - 5y] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4hy + \cancel{y^2} - \cancel{3x} - 3h - \cancel{5y} - \cancel{4xy} - \cancel{y^2} + \cancel{3x} + \cancel{5y}}{h}$$

$$= \frac{4hy - 3h}{h} = 4y - 3$$

Let $h \to 0$

$$\boxed{\frac{\partial z}{\partial x} = f_x(x,y) = 4y - 3}$$

Find and simplify

$$\frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{[4x(y+h) + (y+h)^2 - \cancel{3x} - 5(y+h)] - [4xy + y^2 - \cancel{3x} - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4xh + \cancel{y^2} + 2hy + h^2 - \cancel{5y} - 5h - \cancel{4xy} - \cancel{y^2} + \cancel{5y}}{h}$$

$$\frac{4xh + 2hy + h^2 - 5h}{h}$$

$$4x + 2y + h - 5$$

Let $h \to 0$

$$\boxed{\frac{\partial z}{\partial y} = f_y(x,y) = 4x + 2y - 5}$$

Short-cut:

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

1. Use the derivative rules to find the derivative with respect to *x* (treat y like a constant).

coeff. constant constant

$$z = 4xy + y^2 - 3x - 5y$$

coef coef coef

$$\frac{\partial z}{\partial x} = 4 \cdot 1 \cdot y + 0 - 3 \cdot 1 - 0$$

$$= 4y - 3$$

2. Now use the rules to find the derivative will respect to y (treat x like a constant)

coeff. constant

$$z = 4xy + y^2 - 3x - 5y$$

coef. coef coef.

$$\frac{dz}{dy} = 4x \cdot 1 + 2y - 0 - 5 \cdot 1$$

$$= 4x + 2y - 5$$

Pretend you are skiing on the surface
$z = f(x, y) = 15 - x^2 - y^2$

1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

2. Find their values when
   x = 7 and y = 4

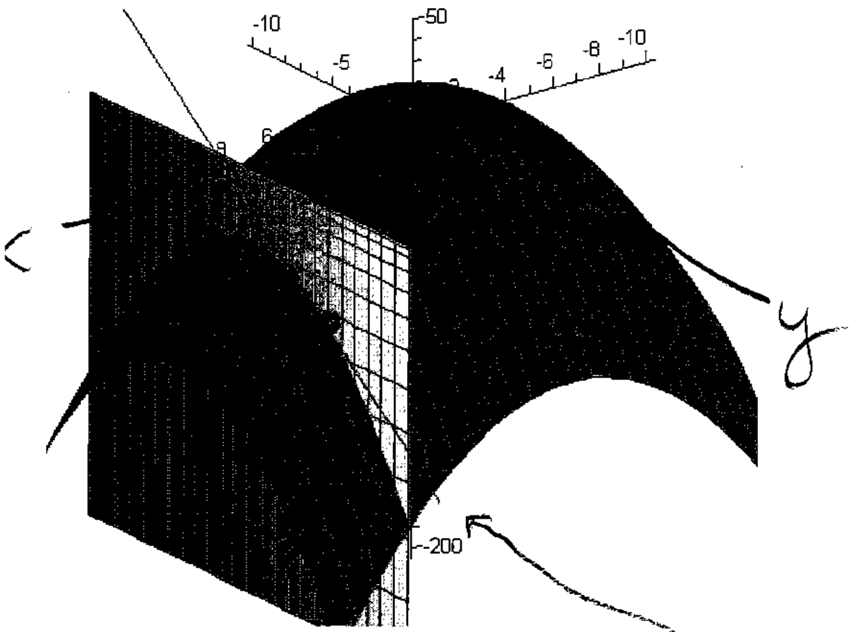
$$\frac{\partial z}{\partial x} = -2x$$

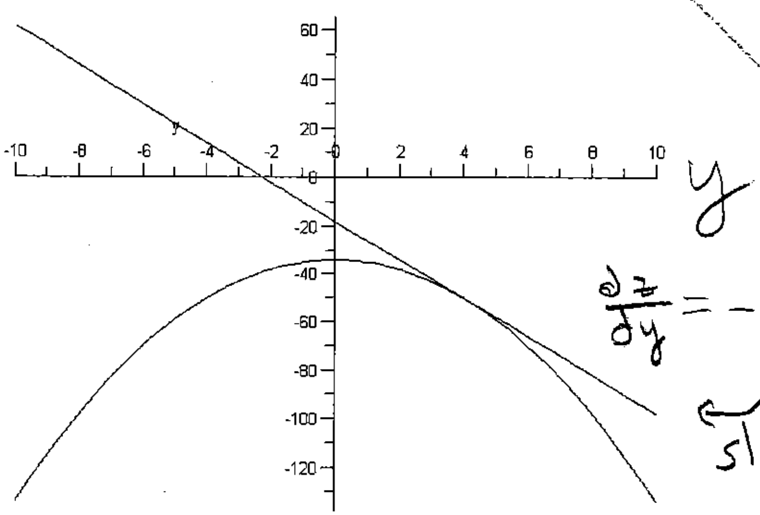
$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial x}(7,4) = -2(7) = -14$$

$$\frac{\partial z}{\partial y}(7,4) = -2(4) = -8$$

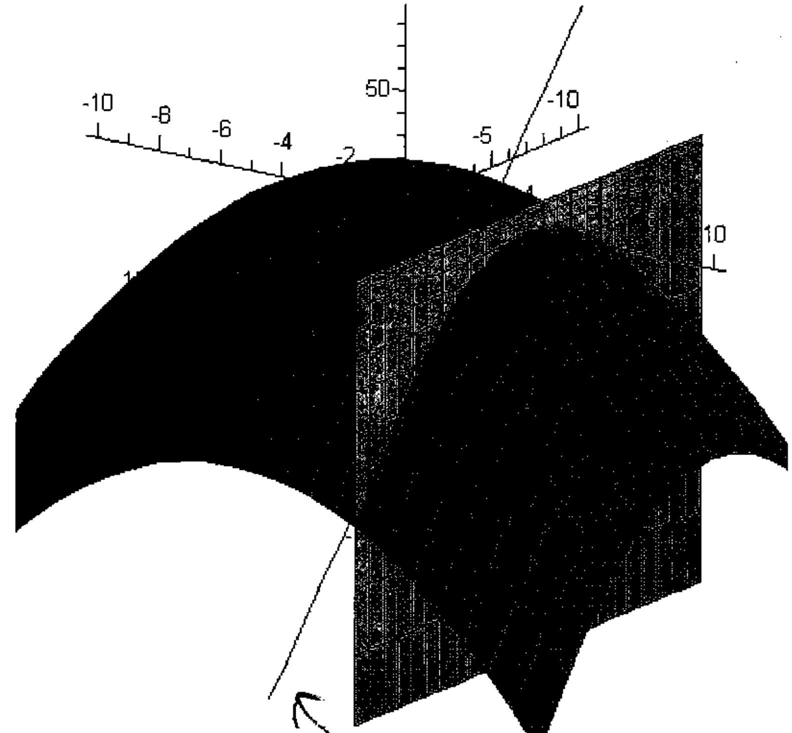
Aside: Graphical Interpretations

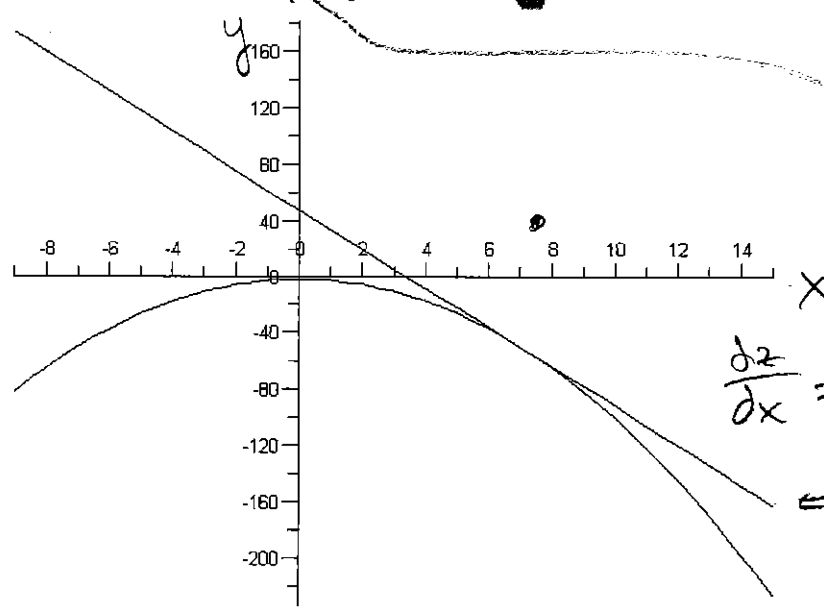




$\frac{\partial z}{\partial y} = -8$

slope





$\frac{\partial z}{\partial x} = -14$

slope

Recall: Before we found the derivative short-cuts, we discussed how:

1. Given a function $y = f(x)$.
2. Simplify the general formula for the slope of the secant from x to $x + h$
$$\frac{f(x+h) - f(x)}{h}$$
3. Let $h \to 0$, to get
$$\frac{dy}{dx} = f'(x) = \text{slope of tangent}$$

Partial Derivatives

For multivariable functions, we are going to fix all the input variables except one (treat them like constants). Then we'll compute the derivative with respect to that one variable function.

Given $z = f(x, y)$

With respect to x as variable: Fix y!
$$\frac{f(x+h, y) - f(x, y)}{h}$$
Let $h \to 0$, to get
$$\frac{\partial z}{\partial x} = f_x(x, y)$$

With respect to y as variable: Fix x!
$$\frac{f(x, y+h) - f(x, y)}{h}$$
Let $h \to 0$, to get
$$\frac{\partial z}{\partial y} = f_y(x, y)$$